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Limit Probability Distributions for an Infinite-Order Phase Transition Model

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The limiting probability distributions for the one-dimensional inhomogeneous spin system considered in a previous paper, which exhibits an infinite-order phase transition, are computed. It turns out that below the critical temperature or in the presence of an external magnetic field, the spins are completely polarized.

KEY WORDS: Inhomogeneous spin model; exactly soluble model; limit probability distributions.

1. INTRODUCTION

In ref. 1 we considered a one-dimensional inhomogeneous Ising chain with very long-range potential and proved that in the absence of external magnetic field, an infinite-order phase transition occurs. The model is a system of spins $\sigma_i = \pm 1$ in an external magnetic field $h \geqslant 0$ with Hamiltonian

$$\mathscr{H}_n[(\sigma)_n; h] = -\sum_{1 \leqslant i \leqslant j \leqslant n} j^{-1}\sigma_i\sigma_j - h \sum_{1 \leqslant i \leqslant n} \sigma_i \tag{1.1}$$

The probability distribution associated to $\mathscr{H}_n$ in the canonical ensemble, defined on the discrete σ-algebra $\mathscr{B}_n = \mathscr{P}(X^n)$, $X \equiv \{-1, 1\}$, is given by

$$p_{n,\beta,h}(A) = \sum_{|k| \leqslant n} f^A_{n,k,h} \Big/ \sum_{|k| \leqslant n} f^{X^n}_{n,k,h}; \qquad A \in \mathscr{B}_n \tag{1.2}$$

where

$$f^A_{n,k,h} = \sum_{A,k} \exp\{-\beta \mathscr{H}_n[(\sigma)_n; h]\} \tag{1.3}$$

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where $\sum_{A,k}$ denotes the sum over all $(\sigma)_n \in A$ with $\sum_{i=1}^{n} \sigma_i = k$. The free energy of the model in the thermodynamic limit is given by[1]

$$\begin{aligned}\mathscr{F}(\beta, h) &\triangleq -\beta^{-1} \lim_{n\to\infty} n^{-1} \log \sum_{|k|\leqslant n} f^{X^n}_{n,k,h} \\ &= \max_{x\in(-1,1)} (\beta x^2 - \tfrac{1}{2}\{(1-|x|)\log[1-y_\beta(|x|)] \\ &\quad + (1+|x|)\log[1+y_\beta(|x|)]\} + \beta h x - \log 2)\end{aligned} \tag{1.4}$$

where y_β: $[0, 1] \mapsto [-1, 1]$ is the unique solution satisfying $|y_\beta| < 1$ for $x < 1$ of the equation[1]

$$(x - y_\beta)\, y'_\beta = \beta x(1 - y_\beta^2) \tag{1.5}$$

for $x \in [0, 1]$. It is shown in ref. 1 that $\mathscr{F}(\cdot, 0)$ is a C^∞ function of $\beta \in \mathbb{R}^+$ and $\mathbb{R}$-analytic in β on $\mathbb{R}^+ \backslash \{\beta_c\}$, where $\beta_c = 1/4$.

In this paper we shall be interested in the behavior of the limiting probability distributions of the model. These are related to the large-n behavior of the $f^A_{n,k,h}$. By (1.3) and (1.1) we have

$$f^A_{n,k,h} = \exp(\beta h k)\, f^A_{n,k,0} \tag{1.6}$$

and thus only the asymptotic behavior of the $f^A_{n,k} \equiv f^A_{n,k,0}$ needs to be studied. In the following we shall identify the event $A \in \mathscr{B}_n$ with all the cylindrical events $A \times X^n$ generated by it and denote $n_A = \inf\{n \in \mathbb{N}$: $A \in \mathscr{B}_n\}$. Let $S_n = \{-n, -n+2,..., n-2, n\}$ and $\Lambda_m = \sum_{n=m}^{\infty} \{n\} \times S_n$. Let also $\Omega(\mathscr{K})$ be the space of real-valued functions on the set $\mathscr{K}$ and let $\mathscr{R}_m$: $\Omega(\Lambda_m) \mapsto \Omega(\Lambda_m)$ be given by

$$\begin{aligned}(\mathscr{R}_m \psi)_{m,k} &= \psi_{m,k} \\ (\mathscr{R}_m \psi)_{n+1,k} &= \exp\left(-\beta \frac{k}{n+1}\right) \psi_{n,k+1} \\ &\quad + \exp\left(\beta \frac{k}{n+1}\right) \psi_{n,k-1} \qquad \text{for} \quad n \geqslant m\end{aligned} \tag{1.7}$$

One obtains from (1.1) and (1.3) the fixed-point equation

$$\mathscr{R}_{n_A} f^A = f^A \tag{1.8}$$

which, viewed as a recursion relation, together with Eq. (1.3) written for $n = n_A$, uniquely determines f^A on Λ_{n_A}.

Let $\mathscr{B} = \sigma(\bigcup_{i\in\mathbb{N}} \mathscr{B}_i)$. We shall need the following convergence result.

Proposition 1. For any $\beta \in \mathbb{R}^+$, $k \in \mathbb{N}$, and $A \in \mathscr{B}_k$, the following limit exists:

$$p_{\beta,0}(A) = \lim_{n \to \infty} p_{n,\beta,0}(A) \tag{1.9}$$

and $p_{\beta,0}(A)$ extends to a probability measure on $\mathscr{B}$.

Proof. Let $A \in \bigcup_{i=0}^{\infty} \mathscr{B}_i$ and for $|j| \leqslant n_0$

$$C_j^A = f_{n_A,j}^A \tag{1.10}$$

Then, by (1.8),

$$f^A = \sum_{j \in S_{n_A}} C_j^A f^{n_A,j} \tag{1.11}$$

where $f^{n_A,j}$ is the solution of Eq. (1.8) with initial condition $f_{n_A,k}^{n_A,j} = \delta_j^k$. Since

$$\sum_{k \in S_n} f_{n,k}^{n_A,j} \Big/ \sum_{k \in S_n} f_{n,k}^{X^n} = \left(\sum_{|k| \leqslant n} C_i^X \left[\sum_{|k| \leqslant n} f_{n,k}^{n_A,i} \Big/ \sum_{|k| \leqslant n} f_{n,k}^{n_A,j} \right] \right)^{-1} \tag{1.12}$$

it suffices to show that

$$T_n^{j,0} \triangleq \sum_{k \in S_n} f_{n,k}^{n_A,j+2} \Big/ \sum_{k \in S_n} f_{n,k}^{n_A,j} \tag{1.13}$$

is an increasing function of n for any $j \geqslant 0$, $j \in S_{n_A}$.

Define $\mathscr{R}^*: \Omega(\Lambda_0) \mapsto \Omega(\Lambda_0)$ by

$$(\mathscr{R}^* \Psi)_{m,k} = \exp\left(\frac{\beta}{n+1}\right)\left[\exp\left(\frac{-\beta k}{n+1}\right) \Psi_{m+1,k-1} + \exp\left(\frac{\beta k}{n+1}\right) \Psi_{m+1,k+1}\right] \tag{1.14}$$

for $m \geqslant 0$ and $k \in S_m$.

Let $n_1 \in \mathbb{N}$, $g \in \Omega(S_{n_1})$, and $H^{n_1}(g)$ be the solution in $\Omega(\Lambda_0 \backslash \Lambda_{n_1})$ of

$$\mathscr{R}^* H^{n_1}(g) = H^{n_1}(g) \tag{1.15a}$$

with

$$H_{n_1,k}^{n_1}(g) = g(k) \tag{1.15b}$$

By Eqs. (1.15) and (1.2) it can be verified through (backward) induction that

$$\sum_{k \in S_n} f^A_{n,k} H^{n_1}_{n,k}(g) = \sum_{k \in S_n} f^A_{n_1,k}\, g(k) \tag{1.16}$$

for any $A \in \mathscr{B}_n$ and n in $\{n_A,..., n_1\}$.

Let

$$H^{n,\lambda} \equiv H^n \left[\cosh\left(\beta\lambda \frac{k}{n+1}\right)\right]$$

Then $H^{n,\lambda}_{m,k} = H^{n,\lambda}_{m,-k}$ for $k \in S_n$ and, as it can be easily shown inductively, $H^{n,\lambda}_{m,k}$ is increasing in $k \geqslant 0$ for fixed m. Let

$$T^{j,\lambda}_n = H^{n,\lambda}_{n_A, j+2} / H^{n,\lambda}_{n_A, j} \tag{1.17}$$

Observe that by (1.16), definition (1.13) is consistent with definition (1.17) for $\lambda = 0$ and also that, by Eqs. (1.15), $T^{j,0}_{n+1} = T^{j,1}_n$. We prove that

$$\frac{\partial}{\partial\lambda} T^{j,\lambda}_n \geqslant 0 \tag{1.18}$$

Taking

$$y^\lambda_{m,k} = \tanh\left[\beta \frac{k}{m+1} + \frac{1}{2} \log \frac{H^{n,\lambda}_{m+1,k+1}}{H^{n,\lambda}_{m+1,k-1}}\right] \tag{1.19}$$

we have $y^\lambda_{m,k} = -y^\lambda_{m,-k}$ for k in S_n and $y^\lambda_{m,k} \in (0, 1)$ for $0 < k \in S_m$, and by Eq. (1.15)

$$y^\lambda_{m-1,k} = \tanh\left\{\beta \frac{k}{m(m+1)} + \frac{1}{2} \log \frac{1 + y^\lambda_{m,k-1}}{1 - y^\lambda_{m,k+1}}\right\} \tag{1.20}$$

for $0 < m \leqslant n_1$ and k in S_{m-1}. Relation (1.18) follows now inductively since

$$\begin{aligned} \frac{\partial}{\partial\lambda} y^\lambda_{n_1,k} = [1 - (y^\lambda_{n_1,k})^2] \Bigg[& \beta \frac{k+1}{n_1+1} \tanh\left(\beta\lambda \frac{k+1}{n_1+1}\right) \\ & - \beta \frac{k-1}{n_1+1} \tanh\left(\beta\lambda \frac{k-1}{n_1+1}\right)\Bigg] \end{aligned} \tag{1.21}$$

and

$$\begin{aligned} \frac{\partial}{\partial\lambda} y^\lambda_{m-1,k} = [1 - (y^\lambda_{m-1,k})^2] \Bigg[& \frac{1}{1 + y^\lambda_{m,k-1}} \frac{\partial}{\partial\lambda} y^\lambda_{m,k-1} \\ & + \frac{1}{1 - y^\lambda_{m,k+1}} \frac{\partial}{\partial\lambda} y^\lambda_{m,k+1}\Bigg] \end{aligned} \tag{1.22}$$

for $(k, n) \in \Lambda_1 \backslash \Lambda_{n_1}$.

2. BEHAVIOR OF THE LIMITING PROBABILITY DISTRIBUTIONS

Due to the very slow decay of the potential and to the absence of translational invariance, the probability distributions show unusual features. For instance, for $h>0$, the spins are completely polarized, in that sense that, with probability one, $\sigma_i = 1$ for any $i \in \mathbb{N}$, although the magnetization $m \triangleq \lim_{n\to\infty}(1/n)\sum_{i=1}^{n}\sigma_i$ is always in $(0, 1)$ for $\beta \in \mathbb{R}^+$; explicitly, it is given by the solution of the equation[1]

$$m = y_\beta[\beta(m+h)] \tag{2.1}$$

Also, $\lim_{h\searrow 0} p_{\beta,h}(\sigma_i = 1) = 1$ for all i and $\beta > \beta_c$, albeit the spontaneous magnetization is always less than 1.

Let z_β: $[-1, 1] \mapsto (-1, 1)$ be the unique solution of (1.5) for $\beta < \beta_c$ satisfying $z_\beta(0) = 0$ and $z'_\beta(0) = \frac{1}{2}[1-(1-4\beta)^{1/2}]$ [see ref. 1 for a detailed study of Eq. (1.5)]. The usual technique[2] shows that $z_\beta(\cdot)$ is a real analytic function in an open interval including $[-1, 1]$.

Define H_β: $[-1, 1] \mapsto \mathbb{R}$ for $\beta < \beta_c$ by

$$H_\beta(x) = \beta x^2 - \tfrac{1}{2}\{(1-x)\log[1-z_\beta(x)] + (1+x)\log[1+z_\beta(x)]\} - \log 2 \tag{2.2}$$

and

$$E_\beta(x) = \int_0^x dt\, \frac{H''_\beta(t)[t^2+1-2tz_\beta(t)] - H''_\beta(0) - H'_\beta(t)}{t - z_\beta(t)} \tag{2.3}$$

We now state our main result.

Theorem 1. Let $n \in \mathbb{N}$, $(\sigma)_n \in X^n$, and $\mu = n^{-1}\sum_{i=1}^{n}\sigma_i$. Then:

(i) For $\beta < \beta_c$,

$$p_{\beta,0}(\{(\sigma)_n\}) = \text{const}_n \cdot p_{n,\beta,0}(\{(\sigma)_n\}) \exp[-nH_\beta(\mu) + E_\beta(\mu)][1 + \mathcal{O}(1/n)] \tag{2.4}$$

where const_n is the normalizing constant.

(ii) For $\beta > \beta_c$,

$$p_{\beta,0}(\{(\sigma)_n\}) = \tfrac{1}{2}\delta(|\mu| - 1) \tag{2.5}$$

(iii) For $h > 0$ and $\beta \in \mathbb{R}^+ \backslash \{\beta_c\}$,

$$p_{\beta,h}(\{(\sigma)_n\}) = \delta(\mu - 1) \tag{2.6}$$

where $\delta(x)$ is the Dirac measure at x.

Remark. We have some evidence (but not a proof) that Eqs. (2.5) and (2.6) are correct for $\beta = \beta_c$ also.

We use the symbols $\mathcal{O}$ and o in the following sense: $g_{n,k} = \mathcal{O}(n^\alpha)$ iff $\sup_{n,k} |g_{n,k} n^{-\alpha}| < \infty$ and $g_{n,k} = o(n^\alpha)$ iff $\lim_{n\to\infty} \sup_{k\in S_n} |g_{n,k} n^{-\alpha}| = 0$.

The proof of this theorem is based on the asymptotic expansion of the statistical weights $f^{i,j}_{n,k}$. Since the obtained expressions are cumbersome, whenever the computations are conducted in an obvious manner or have been already done in a similar situation, only the main steps will be presented.

3. PROOF OF THE THEOREM FOR $\beta > \beta_c$

Let

$$g_{n,k} = f^{n_0, i_0}_{n+n_0, k+i_0} / C_n^{(n+k)/2} \tag{3.1}$$

The factor $C_n^{(n+k)/2}$, which is the $\beta = 0$ solution of Eq. (1.8), is introduced in order to isolate some singularities in the asymptotic expansion. We have

$$\mathscr{L} g = g \tag{3.2a}$$

on Λ_0, where $\mathscr{L}\colon \Omega(\Lambda_0) \mapsto \Omega(\Lambda_0)$ is defined by $(\mathscr{L}\Psi)_{0,k} = \Psi_{0,k}$ and

$$(\mathscr{L}\Psi)_{n+1,k} = \frac{1}{2}\left[\left(1 - \frac{k}{n+1}\right) \exp\left(-\beta \frac{k+i_0}{n+n_0+1}\right) \Psi_{n,k+1} + \left(1 + \frac{k}{n+1}\right) \exp\left(\beta \frac{k+i_0}{n+n_0+1}\right) \Psi_{n,k-1}\right] \tag{3.3}$$

Also,

$$g_{0,k} = \delta_{k,0} \tag{3.2b}$$

$$g_{n,\sigma n} = \exp\left(\beta n - \beta \Delta^\sigma \sum_{j=1}^{n} j^{-1}\right) \tag{3.4}$$

for $\sigma = \pm 1$, where $\Delta^\sigma = n_0 - \sigma i_0$. For the proof of the theorem at $\beta > \beta_c$, we shall find a positive function $\tilde{g}$ such that $\log(g/\tilde{g}) = \mathcal{O}(1)$. The latter condition is satisfied in case for some $\varepsilon > 0$

$$\tilde{g}_{n+1,k} = [1 + \mathcal{O}(n^{-1-\varepsilon})](\mathscr{L}\tilde{g})_{n,k} \tag{3.5}$$

for all $n, k \in \Lambda_0$.

A few words on how to obtain $\tilde{g}$ are in order. We consider here $\beta > \beta_c$; for $\beta < \beta_c$ the technique is similar and will not be repeated. In view of (1.4) we should have for large n, $g_{n,k} \sim \exp[n\Gamma_\beta(k/n)]$, where Γ_β is related to the thermodynamic potential and is smooth for $x \neq 0$. Taking $g_{n,k} = \exp[n\Gamma_\beta(k/n)]\, g_{n,k}^{(1)}$, we obtain with the aid of Eqs. (1.8) and (3.1) a recursion relation for $g_{n,k}^{(1)}$ of the form

$$A_{n,k+1}\, g_{n,k+1}^{(1)} + A_{n,k-1}\, g_{n,k-1}^{(1)} = g_{n+1,k}^{(1)} \tag{3.6}$$

In order to obtain minimal growth with n of $\sup_{k \in S_n} |\log g_{n,k}^{(1)}|$, the coefficients in (3.6) should satisfy the condition

$$A_{n,k+1} + A_{n,k-1} = 1 + o(1) \tag{3.7}$$

Condition (3.7) leads to the following equation for Γ_β:

$$\tfrac{1}{2}(1-x)\exp[-\beta x + (x+1)\,\Gamma'_\beta - \Gamma_\beta] + \tfrac{1}{2}(1+x)\exp[\beta x + (x-1)\,\Gamma'_\beta - \Gamma_\beta] = 1 \tag{3.8a}$$

$$\Gamma_\beta(\pm 1) = 1 \tag{3.8b}$$

The function $\Gamma_\beta^{(0)}$ defined on the domain of y_β, i.e., on $(-\varepsilon_\beta, 1]$ with $\varepsilon_\beta > 0$, by

$$\Gamma_\beta^{(0)} = -\frac{1}{2}\left[(1-x)\log\frac{1-y_\beta}{1-x} + (1+x)\log\frac{1+y_\beta}{1+x}\right] \tag{3.9}$$

satisfies Eqs. (3.8a) on $(-\varepsilon_\beta, 1)$, $\Gamma_\beta^{(0)}(1) = 1$, and, moreover,

$$\frac{1}{2}(1 \mp x)\exp\left[\mp\beta x + (x \pm 1)\frac{d}{dx}\Gamma_\beta^{(0)} - \Gamma_\beta^{(0)}\right] = \frac{1}{2}(1 \mp y_\beta) \tag{3.10}$$

For the next step we repeat the above construction in terms of $g_{n,k}^{(1)} \triangleq \exp[\eta(k,n)]\, g_{n,k}^{(2)}$. Assuming that $\eta(\cdot,\cdot)$ is smooth on some domain of the variables n, k (thought of as continuous variables) and neglecting some higher-order terms, we obtain for

$$\psi^+(n, k/n) \triangleq \beta\Delta^+ \log(n+1) + \eta(n,k) \tag{3.11}$$

the equation in $x = k/n \in (-\varepsilon_\beta, 1)$,

$$\frac{d}{dx}\psi^+ = \frac{x^2+1-2xy_\beta}{2(y_\beta - x)}\frac{d^2}{dx^2}\Gamma_\beta^{(0)} + n_0\frac{1-xy_\beta}{y_\beta - x} + i_0\frac{y_\beta - 1}{y_\beta - x} \triangleq H_\beta^+ \tag{3.12}$$

This perturbative process could be (at least in principle) continued, but, in order to obtain thermodynamic information, a finite number of steps is sufficient. In our case, the first step is sufficient to obtain the free energy and the second for the limit probability distributions for $\beta > \beta_c$, whereas for $\beta < \beta_c$, some additional information on $g^{(2)}_{n,k}$ is needed. The perturbative method should work for other models, too, provided a convenient asymptotic relationship among the relevant variables could be established.

Returning to the proof, we have to define properly the functions obtained above, define $\tilde{g}_{n,k}$ on the whole Λ_0, and verify (3.5).

It follows easily from the proof given in ref. 1 for the analyticity of y_β that $y_\beta(\cdot)$ is an analytic function in the complex variable x in a neighborhood of $x=1$ and thus real analytic on $(-\varepsilon_\beta, 1+\varepsilon_\beta)$ for some positive ε_β. Since $y'_\beta(1) = 1+2\beta$, $\Gamma^{(0)}_\beta(\cdot)$ and $H_\beta(\cdot)$ given by (3.12) are also real analytic on the same interval. Let

$$\Gamma_\beta(x) = \begin{cases} \Gamma^{(0)}_\beta(x) & \text{for} \quad x \in (-\varepsilon_\beta, 1+\varepsilon_\beta) \\ -\infty & \text{otherwise} \end{cases} \tag{3.13}$$

and, for $\sigma = \pm 1$,

$$\begin{aligned} \psi^\sigma_{n_0, i_0}(x) = \int_\sigma^x dt \left\{ \frac{1}{2} \frac{t^2+1-2ty_\beta(\sigma t)}{t-y_\beta(\sigma t)} \Gamma''_\beta(t) \right. \\ \left. + \frac{n_0[1-ty_\beta(\sigma t)]}{y_\beta(\sigma t)-t} + \sigma \frac{k_0[y_\beta(\sigma t)+\sigma]}{y_\beta(\sigma t)-t} \right\} \\ -\beta\Delta^\sigma \log(n+1) \end{aligned} \tag{3.14}$$

and

$$\tilde{g}_{n,k} = \sum_{\sigma = \pm 1} \exp[n\Gamma_\beta(\sigma x) + \psi^\sigma_{n_0, i_0}(x)] \tag{3.15}$$

where $x = k/n$.

Lemma 3.1. For $\beta > \beta_c$ and $h=0$, for some $C_{1,2} \in \mathbb{R}^+$,

$$C_1 \tilde{g}_{n,k} < g_{n,k} < C_2 \tilde{g}_{n,k} \tag{3.16}$$

This lemma is proved inductively, using Eq. (3.5). The proof of Eq. (3.5) is most easily performed in the following steps:

1. The case $k = \pm n$ follows from (3.15), (3.14), (3.10b), and (3.4).

2. For $\sigma k/n \in (-\varepsilon_\beta, 1)$ the $\sigma = 1$ and $\sigma = -1$ terms in the sum (3.6) satisfy (3.5) [and then their sum fulfills (3.5) on $(-\varepsilon_\beta, \varepsilon_\beta)$], as can be checked using the series expansion of the exponent, the analyticity of the functions, and Eqs. (3.8)–(3.11) (see ref. 1 for details in a similar case).

3. For $k/n > \varepsilon_\beta/2$, the $\sigma = -1$ term in Eq. (3.15) is $\exp(-|\text{const}| \cdot n)$ times smaller than the $\sigma = 1$ term and can thus be discarded in this region. Similarly, the $\sigma = 1$ term can be dropped from the sum, for $k/n < -\varepsilon_\beta/2$. ∎

In conclusion, with

$$\bar{f}^{n_0,i_0}_{n,k,h} = \exp(\beta hk)\, C_{n-n_0}^{(n-n_0+k-i_0)/2} \times \sum_{\sigma=\pm 1} \exp\left[(n-n_0)\,\Gamma_\beta\left(\sigma\,\frac{k-i_0}{n-n_0+1}\right) + \psi^\sigma_{\beta,n_0,i_0}\left(\frac{k-i_0}{n-n_0+1}\right)\right] \tag{3.17}$$

we have

$$\text{const} < f^{n_0,i_0}_h / \bar{f}^{n_0,i_0}_h < \text{const}' \tag{3.18}$$

on Λ_0.

Remark. In fact, it can be shown, using estimates on $f^{n_0,i_0}_{n,k,h}/\bar{f}^{n_0,i_0}_{n,k,h} - f^{n_0,i_0}_{n,n,h}/\bar{f}^{n_0,i_0}_{n,n,h}$, much in the same way in which (4.27) below is obtained, that

$$\lim_{n\to\infty} f^{n_0,i_0}_{n,k,h}/\bar{f}^{n_0,i_0}_{n,k,h} = 1$$

if $k/n \to x \neq 0$.

In view of (3.18) it follows, in a straightforward manner, that

$$p_{\beta,n_A,0}(\{(\sigma)_{n_A}\}) = \mathcal{O}\left(\exp\left[-\beta \log n\left(n_A - \left|\sum_{|i|\leqslant n_A} \sigma_i\right|\right)\right]\right) \tag{3.19}$$

$$p_{\beta,n_A,h}(\{(\sigma)_{n_A}\}) = \mathcal{O}\left(\exp\left[-\beta \log n\left(n_A - \sum_{|i|\leqslant n_A} \sigma_i\right)\right]\right) \tag{3.20}$$

proving Theorem 1 for $\beta > \beta_c$.

4. THE LIMIT PROBABILITY DISTRIBUTIONS FOR $\beta < \beta_c$

Let $g_0 = y'_\beta(0) > 1/2$ and $\tilde{g}: \Lambda_0 \mapsto \mathbb{R}^+$ be given by

$$\tilde{g}_{n,k} = \exp[n\Gamma_\beta(k/n) + \psi_\alpha(k, n)] \tag{4.1}$$

with

$$\psi_\alpha(k, n) = \int_{n^{g_0-1}}^1 dx\, w_0(x) + \sum_{\sigma=\pm 1} P_\alpha(\sigma k, n) \times\left[\int_x^{n^{g_0-1}} dx\, w_0(x) + \int_x^1 dx\, w^\sigma(x) - \beta(n_0 - \sigma i_0) \sum_{j=1}^{n+1} j^{-1}\right] \tag{4.2}$$

$$w_0(x) = \tfrac{1}{2}\Gamma''_\beta(x)[x - y_\beta(x)]^{-1}\, [x^2 + 1 - 2xy_\beta(x)] \tag{4.3}$$

$$w^\sigma(x) = \{n_0[1 - xy_\beta(x)] - \sigma i_0[1 - y_\beta(x)]\}[x - y_\beta(x)]^{-1} \tag{4.4}$$

and

$$P_\alpha(k, n) = \begin{cases} k^\alpha n^{-\alpha g_0}(1 + kn^{-g_0})^{-\alpha} & \text{for} \quad k > n^{g_0 - 2/\alpha} \\ 0 & \text{otherwise} \end{cases} \tag{4.5}$$

Define also

$$\tilde{H}_{n,k} = \exp[-nH_\beta(k/n) + E_\beta(k/n) + \beta \log n] \tag{4.6}$$

Then, the following estimates hold.

Lemma 4.1. (i) If $\alpha > 2(g_0 - 1/2)^{-1}$, then, for some $\vartheta_\beta > 1$,

$$(\mathscr{L}\tilde{g})_{n,k} = [1 + \mathcal{O}(n^{-\vartheta_\beta})]\, \tilde{g}_{n+1,k} \tag{4.7}$$

(ii)

$$(\mathscr{R}^* \tilde{H})_{m,k} = [1 + \mathcal{O}(m^{-2})]\, \tilde{H}_{m,k} \tag{4.8}$$

The proof follows essentially the scheme of ref. 1 and uses again Eqs. (3.8)–(3.12). For the proof of (i), however, the following remarks are useful. For small positive ε_1 and large n, P_α $[(1 - P_\alpha)]$ is negligible in the region $|k/n| < n^{g_0 - \varepsilon_1}$ $[|k| > n^{g_0 + \varepsilon_1}$, respectively$]$. Now, for $n^{g_0 - \varepsilon_1} \leqslant |k| \leqslant n^{g_0 + \varepsilon_1}$, after the appropriate series expansions, the expression of $\tilde{g}_n^{-1}(\mathscr{L}\tilde{g})_n$ is given, to the required degree of accuracy, by a convex combination of the terms corresponding to $P_\alpha = 0$ and $P_\alpha = 1$.

It follows from (4.7) that for some $C_{1,2} \in \mathbb{R}^+$,

$$C_1\, \tilde{g} < g < C_2\, \tilde{g} \tag{4.9}$$

on Λ_0 and then by (4.1), (4.9), and (3.1), that for any $\varepsilon > 0$ and $A, A' \in \bigcup_{i \in \mathbb{N}} \mathscr{B}_i$,

$$\lim_{n \to \infty} \left(\sum_{k \in S_n} f^A_{n,k} \Big/ \sum_{k \in S_n} f^{A'}_{n,k} \right)$$
$$= \lim_{n \to \infty} \left(\sum_{|k| < n^{1/2+\varepsilon}} f^A_{n,k} \Big/ \sum_{|k| < n^{1/2+\varepsilon}} f^{A'}_{n,k} \right) \in \mathbb{R}^+ \tag{4.10}$$

Relation (3.9) is sufficient for completing the proof of Theorem 1(iii), but in order to obtain estimates for the limits in (4.10) ($h = 0$ case), we need to relate the constants appearing in Eq. (4.9) to the initial conditions on f^A [i.e., to (1.3) written for $n = n_A$]. The recursion (3.2) is linear and thus the study of its asymptotic regime alone cannot yield the needed relationship. We shall instead refer to (1.16) and note that

$$\lim_{n\to\infty}\left(\sum_{k\in S_n} f_{n,k}^{n_0,i_0}\Big/\sum_{k\in S_n} f_{n,k}^{n_1,i_1}\right)=\lim_{n\to\infty}\left(H_{n_0,i_0}^n(1)/H_{n_1,i_1}^n(1)\right) \tag{4.11}$$

and show that the limit on the rhs of (4.11) is equal to

$$\lim_{n\to\infty}\left(H_{n_0,i_0}^n(\tilde{H}_n)/H_{n_1,i_1}^n(\tilde{H}_n)\right) \tag{4.12}$$

for which, as a consequence of Lemma 4.1(ii), $\tilde{H}_{n_0,i_0}/\tilde{H}_{n_1,i_1}$ is an estimate with the required degree of accuracy. Let

$$\chi_{n,k}=g_{n,k}/\tilde{g}_{n,k} \tag{4.13}$$

Using (3.2), one obtains a recursion relation for the χ_n of the form

$$\chi_{n+1,k}=\tfrac{1}{2}(1-\tilde{y}_{n+1,k})\,\chi_{n,k+1}+\tfrac{1}{2}(1+\tilde{y}_{n+1,k})\,\chi_{n,k-1}[1+\mathcal{O}(n^{-1-\eta_\beta})] \tag{4.14}$$

for some positive η_β and where

$$(1-\tilde{y}_{n+1,k})=\left[1-y_\beta\left(\frac{k}{n+1}\right)\right][1+\mathcal{O}(n^{-1})] \tag{4.15}$$

and

$$\tilde{y}_{n,k+1}-\tilde{y}_{n,k-1}=y_\beta\left(\frac{k+1}{n}\right)-y_\beta\left(\frac{k-1}{n}\right)+\mathcal{O}(n^{-1-\eta_\beta}) \tag{4.16}$$

Lemma 4.2.

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\sup_{|k|<n^{g_0-\varepsilon}}|\chi_{n,k}-\chi_{n,0}|=0 \tag{4.17}$$

Since, as a consequence of Lemma 4.1(i), χ is uniformly bounded on Λ_0, it suffices to prove that Eq. (4.17) holds for the solution of the recursion

$$\tilde{\chi}_{n+1,k}=\tfrac{1}{2}(1-\tilde{y}_{n+1,k})\,\tilde{\chi}_{n,k+1}+\tfrac{1}{2}(1+\tilde{y}_{n+1,k})\,\tilde{\chi}_{n,k-1} \tag{4.18}$$

with initial conditions $\tilde{\chi}_{n_1}=\chi_{n_1}$, for all large enough n_1. Indeed, in this assumption, since by (4.14) and (4.18), for some positive ε_β

$$\tilde{\chi}_{n,k}=\exp[\mathcal{O}(n_1^{-\varepsilon_\beta})]\,\chi_{n,k} \tag{4.19}$$

we have

$$|\chi_{n,k}-\chi_{n,0}| \leqslant \exp(\text{const}\cdot n_1^{-\varepsilon_\beta})\,|\tilde{\chi}_{n,k}-\tilde{\chi}_{n,0}| + \sup_{n,k}|\tilde{\chi}_{n,k}|\cdot\text{const}'\cdot n^{-\varepsilon_\beta} \tag{4.20}$$

Taking

$$D_{n,k} \triangleq n^{g_0}(\tilde{\chi}_{n,k+1}-\tilde{\chi}_{n,k-1}) \tag{4.21}$$

in Eq. (4.18), we obtain

$$D_{n+1,k}=\frac{1}{2}\left[1+\frac{g_0}{n+1}+\mathcal{O}(n^{-2})\right][(1-\tilde{y}_{n+1,k+1})\,D_{n,k+1} + (1+\tilde{y}_{n+1,k-1})\,D_{n,k-1}] \tag{4.22}$$

Let

$$C_n=\sup_{k\in S_n} D_{n,k} \tag{4.23}$$

By (4.16),

$$D_{n+1,k}=\frac{1}{2}\,C_n\left(1+\frac{g_0}{n+1}\right)\left[2+y_\beta\left(\frac{k-1}{n}\right)-y_\beta\left(\frac{k+1}{n}\right)+\mathcal{O}(n^{-1-\eta_\beta})\right] \tag{4.24}$$

and since $y'_\beta(x)\geqslant g_0 x$ on $[0,1]$,[1] it follows that

$$D_{n+1,k}<C_n[1+\mathcal{O}(n^{-1-\eta_\beta})] \tag{4.25}$$

and thus $D<\text{const}$ on Λ_{n_1}. The inequality $D>-\text{const}$ is obtained in the same manner and thus

$$|\tilde{\chi}_{n,k}-\tilde{\chi}_{n,0}|<\text{const}\cdot kn^{-g_0} \tag{4.26}$$

Summing up, we have, for large n,

$$f_{n,k}^{n_0,i_0}=C_{n-n_0}^{(n-n_0+k-i_0)/2}\tilde{g}_{n-n_0,k-i_0}\cdot(\chi_{n,0}^{n_0,i_0}+\gamma_{n,k}) \tag{4.27}$$

with

$$\sup_{n,k}|\gamma_{n,k}|<\infty \tag{4.28}$$

and

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\sup_{|k|<n^{g_0-\varepsilon}}|\gamma_{n,k}|=0 \tag{4.29}$$

Using (4.1), (4.6), and (4.27)–(4.29) and the properties of Eq. (1.5), it follows, in a straightforward manner, that

$$\lim_{n \to \infty} \left(\sum_{k \in S_n} f_{n,k}^{n_0, i_0} \Big/ \sum_{k \in S_n} f_{n,k}^{n_1, i_1} \right)$$
$$= \lim_{n \to \infty} (\chi_{n,0}^{n_0, i_0} / \chi_{n,0}^{n_1, i_1})$$
$$= \lim_{n \to \infty} \left(\sum_{k \in S_n} f_{n,k}^{n_0, i_0} \tilde{H}_{n,k} \Big/ \sum_{k \in S_n} f_{n,k}^{n_1, i_1} \tilde{H}_{n,k} \right) \tag{4.30}$$

Relation (4.30), together with (1.16) and Lemma 4.1(ii), completes the proof of Theorem 1.

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